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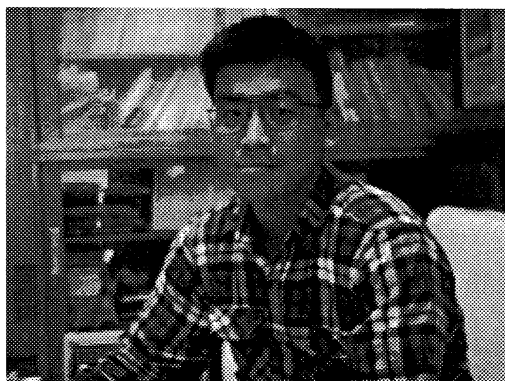
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BRANCHING OF SINGULARITIES FOR SOME
THIRD ORDER MICROHYPERBOLIC OPERATORS
MULTIPLY CHARACTERISTIC AT $x_1 = 0$

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§1 INTRODUCTION

In this brief report, we consider the branching of the support of microfunction solutions to a microhyperbolic equation of third order, triply characteristic over the initial surface $x_1 = 0$. Branching of singularities has been studied by many authors. Alinhac and Taniguchi-Tozaki made researches into second order hyperbolic operators in the C^∞ -category. Hanges and Oaku treated operators of the form $x_1 D_1 -$ (lower), in the C^∞ - and C^ω -categories respectively. Amano-Nakamura studied an operator of arbitrary order and reduced the problem of branching of C^∞ -singularities to that of Stokes phenomena.

§2 RESULTS

Let

$$P(x, D) = D_1^3 - x_1^2 D_n^2 D_1 + 2(a-b) D_n D_1 + \{2(a+b) - 3\} x_1 D_n^2 + \sum_{l=0}^{\text{finite}} \alpha_{-l}(x_1^2, x', D') x_1^{l+1} D_1^l$$

be a microdifferential operator defined near a point p in $\{(x, i\xi) \in iT^*\mathbb{R}^n; x_1 = 0, \xi_n > 0\}$. Here we assume that $\text{ord}\alpha_{-l} \leq -l - 1$ and that α_{-l} is an polynomial in $t = \frac{1}{2}x_1^2$ and x_n .

Here we write $x = (x_1, x_2, \dots, x_n) = (x_1, x')$, $D_j = \frac{\partial}{\partial x_j}$ ($1 \leq j \leq n$),

a and b are constants such that

$$a \notin \mathbb{Z}, a + b \notin \frac{1}{2} + \mathbb{Z}, b \notin \mathbb{Z}.$$

$\sigma(P)$, the principal symbol of P , has the factorization

$$\sigma(P) = (\xi_1 - x_1\xi_n)\xi_1(\xi_1 + x_1\xi_n).$$

Hence P is microhyperbolic and triply characteristic at $x_1 = 0$. In $x_1 \neq 0$, it is simply characteristic and we can apply a propagation theorem of SKK. That is, the support of a solution to P is a union of bicharacteristic strips, each of which is parametrized by x_1 . Let b_j^\pm be the half bicharacteristic strip in $\pm x_1 > 0$ issuing from p , contained in $\xi_1 = x_1\xi_n, 0, -x_1\xi_n$ for $j = 1, 2, 3$ respectively. A microfunction solution u to P , defined in the intersection of a neighborhood of p and $\{(x, i\xi dx); x_1 > 0\}$, is said to be j -pure if $u = 0$ on b_k^+ ($k \neq j$) and $u \neq 0$ on b_j . Remark that according to the general theory due to Kashiwara-Kawai on microhyperbolic operators, we have the isomorphism

$$(\Gamma_{\{x_1 > 0\}}\mathcal{C}_M^P)_p \simeq \mathcal{C}_{M,p}^P$$

where $M = \mathbb{R}_x^n$ and \mathcal{C}_M^P is the solution sheaf. It means unique extendability of solutions across $x_1 = 0$.

Now we pose the following problem.

PROBLEM 1 (branching of singularities).

What is the support of the extension of a j -pure solution?

Let $N = \{x; x_1 = 0\} \subset \mathbb{R}^n$, ρ be the pull-back $N \times_M iT^*M \rightarrow iT^*N$ and $p' = \rho(p)$.

We have the boundary value (iso)morphism

$$\begin{aligned} b.v. : (\Gamma_{\{x_1 > 0\}}\mathcal{C}_M^P)_p &\xrightarrow{\sim} \bigoplus^3 \mathcal{C}_{N,p'} \\ u &\mapsto (D_1^k u(+0, x'))_{k=1,2,3}. \end{aligned}$$

Our next problem is

PROBLEM 2 (boundary value problem with purity). *Let $f(x')$ be an element of $\mathcal{C}_{N,p'}$. Consider*

$$(*) \begin{cases} Pu = 0 & \text{in } x_1 > 0 \\ u \text{ is } j\text{-pure} \\ u(+0, x') = f(x') \end{cases}$$

We give answers to the two problems above. First we have

ANSWER TO PROBLEM 2. *(*) is uniquely solvable for a generic (a, b) . More precisely, there is a holomorphic function G in $\{(a, b) \in \mathbb{C}^2; a \notin \mathbb{Z}, a+b \notin \frac{1}{2} + \mathbb{Z}, b \notin \mathbb{Z}\}$, not vanishing identically, such that (*) is uniquely solvable if $G(a, b) \neq 0$.*

REMARK

G can be written explicitly in terms of integrals which resemble that defining Beta function.

Next, we have

ANSWER TO PROBLEM 1(GENERIC CASE).

For a generic (a, b) , we have the following: If a solution u is pure, its extension across $x_1 = 0$ has three branches: $u \neq 0$ on each b_j^- ($j = 1, 2, 3$).

If $\alpha_{-l} = 0$ for all l , we can consider the case $(a, b) \in \mathbb{N} \times \mathbb{N}, \mathbb{N} = \{1, 2, 3, \dots\}$.

This non-generic case is interesting in that we encounter a different kind of branching phenomenon.

ANSWER TO PROBLEM 1 (NON-GENERIC CASE).

Under the condition above, we have: (1) If a solution u is 1-pure, then $u \neq 0$ on b_1^- , $u \neq 0$ on b_2^- and $u = 0$ on b_3^- .

(2) If u is 2-pure, $u = 0$ on $b_1^- \cup b_3^-$ and $u \neq 0$ on b_2^- .

(3) If u is 3-pure, $u = 0$ on b_1^- , $u \neq 0$ on b_2^- and $u \neq 0$ on b_3^- .

Roughly speaking, it means that b_2^- is "privileged". Under other conditions, it happens that another half-bicharacteristic strip is privileged in a similar sense. In fact, we can prove that if $b \in \mathbb{N}$ and $a + b \in \frac{3}{2} - \mathbb{N}$, then b_1^- is privileged and that if $a \in \mathbb{N}$ and $a + b \in \frac{3}{2} - \mathbb{N}$, then b_3^- is.

§3 SKETCH OF THE PROOF

The problems are solved by constructing a "j-pure fundamental solution". That is, we construct a morphism

$$E_j : \mathcal{C}_{N,p'} \rightarrow (\Gamma_{\{x_1 > 0\}} \mathcal{C}_M^P)_p$$

$$f(x') \mapsto (E_j f)(x)$$

such that $E_j f$ is a j-pure solution. Once we know the boundary values

$$D_1^k(E_j f)(+0, x') \quad (k = 0, 1, 2, j = 1, 2, 3),$$

we immediately obtain the results in the previous section. We perform the change of variables $t = \frac{1}{2}x_1^2$ for $x_1^3 P$, and apply the quantized Legendre transform with respect to (t, x') . Then the operator to be considered is

$$Q(\zeta, x', \partial_\zeta, D_{x'}) = J(\zeta, \partial_\zeta) + J'(\zeta, x', \partial_\zeta, D'_x)$$

where

$$J(\zeta, \partial_\zeta) = (\zeta^3 + \zeta) \partial_\zeta^3 + \left\{ \frac{15}{2} \zeta^2 - i(a-b)\zeta + a + b + \frac{3}{2} \right\} \partial_\zeta^2 + \{12\zeta - 2i(a-b)\} \partial_\zeta + 3$$

$$J' = \sum_{m=2}^{\text{finite } m-2} \sum_{j=0} \alpha_{m,j}(x', D') \zeta^j \partial_\zeta^m \in \mathcal{E}(-1)$$

$$\text{ord} \alpha_{m,j} \leq -m - 1.$$

J is an ordinary differential operator of Fuchs type called Jordan-Pochhammer hypergeometric operator. Its solutions have Euler integral representation. A suitably chosen path gives us a solution with a desired singularity. Such a solution roughly corresponds to a j-pure solution. Of course we have to deal with the perturbation J' . This is performed by successive approximation. We construct a microdifferential operator $\tilde{E}_j(\zeta, x', D')$ of order 0 such that :

\tilde{E}_j is defined in (a neighborhood in \mathbb{C}_ζ of $\{\zeta; \operatorname{Re} \zeta \geq 0, \zeta \neq a_j\} \times$ (a conic neighborhood in iT^*N of p'). Here $a_j = i, 0, -i$ if $j = 1, 2, 3$ respectively.

$$(\tilde{E}_j f)(\zeta, x') \in \mathcal{CO}_+^\infty \quad \text{for all } f \in \mathcal{C}_{N,p'}$$

$$\tilde{E}_j \in \zeta^{-1} \mathcal{E}(0) + \zeta^{-\frac{3}{2}} \mathcal{E}(0) \quad \text{at } \zeta = \infty$$

$$Q(\zeta, x', \partial_\zeta, D'_x) \tilde{E}_j(\zeta, x', D') = 0$$

Here $\partial_\zeta = [D_\zeta, \bullet]$. Obviously, for any $f(x') \in \mathcal{C}_{N,p'}$, we have

$$Q(\zeta, x', D_\zeta, D_{x'},) [\tilde{E}_j(\zeta, x', D') f(x')] = 0$$

According to [Kat], $\tilde{E}_j f$ defines a j -pure solution, which is the definition of $(E_j f)(x)$. Its boundary values are calculated from the expansion coefficients of Q at $\zeta = \infty$.

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